Curso de Relatividad General: Métrica Toroidal

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$$\mathrm{d}s^2 = \underbrace{a^2}_{\mathbf{e}_1\cdot\mathbf{e}_1}\mathrm{d}\alpha^2 + \underbrace{(b+a\cos\alpha)^2}_{\mathbf{e}_2\cdot\mathbf{e}_2}\mathrm{d}\theta^2 \tag{1}$$

1 Símbolos de Christoffel

Utilizando el truco descrito por Javier en el capítulo 32 de la serie tenemos, haciendo $x^1 \equiv \alpha$ i $x^2 \equiv \theta$:

$$\partial_1(\mathbf{e}_1\cdot\mathbf{e}_1) = \begin{cases} 2(\partial_1\mathbf{e}_1)\cdot\mathbf{e}_1 = 2\Gamma^1_{11}(\mathbf{e}_1\cdot\mathbf{e}_1) = 2a^2\Gamma^1_{11} \\ \frac{\partial a^2}{\partial\alpha} = 0 \end{cases} \Longrightarrow \Gamma^1_{11} = 0 \tag{2}$$

$$\partial_2(\mathbf{e}_1\cdot\mathbf{e}_1) = \begin{cases} 2(\partial_2\mathbf{e}_1)\cdot\mathbf{e}_1 = 2\Gamma^1_{21}(\mathbf{e}_1\cdot\mathbf{e}_1) = 2a^2\Gamma^1_{21} \\ \frac{\partial a^2}{\partial\theta} = 0 \end{cases} \Longrightarrow \Gamma^1_{21} = \Gamma^1_{12} = 0 \tag{3}$$

$$\partial_1(\mathbf{e}_2\cdot\mathbf{e}_2) = \begin{cases} 2(\partial_1\mathbf{e}_2)\cdot\mathbf{e}_2 = 2\Gamma^2_{12}(\mathbf{e}_2\cdot\mathbf{e}_2) = 2(b+a\cos\alpha)^2\Gamma^2_{12} \\ \frac{\partial(b+a\cos\alpha)^2}{\partial\alpha} = -2a(b+a\cos\alpha)\sin\alpha \end{cases} \Longrightarrow \Gamma^2_{12} = \Gamma^2_{21} = -\frac{a\sin\alpha}{b+a\cos\alpha} \tag{4}$$

$$\partial_2(\mathbf{e}_2\cdot\mathbf{e}_2) = \begin{cases} 2(\partial_2\mathbf{e}_2)\cdot\mathbf{e}_2 = 2\Gamma^2_{22}(\mathbf{e}_2\cdot\mathbf{e}_2) = 2(b+a\cos\alpha)^2\Gamma^2_{22} \\ \frac{\partial(b+a\cos\alpha)^2}{\partial\theta} = 0 \end{cases} \Longrightarrow \Gamma^2_{22} = 0 \tag{5}$$

$$\partial_1(\mathbf{e}_1\cdot\mathbf{e}_2) = (\partial_1\mathbf{e}_1)\cdot\mathbf{e}_2 + \mathbf{e}_1\cdot(\partial_1\mathbf{e}_2) = (b+a\cos\alpha)^2\Gamma^2_{11} + a^2\underbrace{\Gamma^1_{12}}_{0} = 0 \Longrightarrow \Gamma^2_{11} = 0 \tag{6}$$

$$\begin{aligned} \partial_2(\mathbf{e}_1\cdot\mathbf{e}_2) = (\partial_2\mathbf{e}_1)\cdot\mathbf{e}_2 + \mathbf{e}_1\cdot(\partial_2\mathbf{e}_2) = (b+a\cos\alpha)^2\Gamma^2_{21} + a^2\Gamma^1_{22} = 0 \\ \Longrightarrow \Gamma^1_{22} = -\left(\frac{b+a\cos\alpha}{a}\right)^2\Gamma^2_{21} = \frac{(b+a\cos\alpha)\sin\alpha}{a} \end{aligned} \tag{7}$$

Notemos que para que esto sea cierto debemos imponer que $a \neq 0$ (es decir, el toro debe tener cierto grosor) y $b + a\cos\alpha \neq 0$, que si queremos que sea cierto para cualquier α debe cumplirse $b > a$ (es decir, debe ser un toro de anillo).
En resumen los únicos símbolos diferentes de cero son:

$$\Gamma^2_{12} = \Gamma^2_{21} = -\frac{a\sin\alpha}{b+a\cos\alpha}$$

$$\Gamma^1_{22} = \frac{b+a\cos\alpha}{a}\sin\alpha$$

2 Tensor de Riemann

Para calcular el tensor de Riemann utilicemos que

$$(\partial_\alpha \partial_\beta - \partial_\beta \partial_\alpha)\, \mathbf{e}_\gamma = R^\sigma{}_{\gamma\alpha\beta} \mathbf{e}_\sigma$$

Se puede ver inmediatamente que es un tensor antisimétrico respeto a $\alpha \leftrightarrow \beta$, por lo tanto si $\alpha = \beta$ el tensor vale 0. Para calcular las otras calculemos primero lo siguiente:

$$\partial_1 \Gamma^2_{21} = \frac{\partial \Gamma^2_{12}}{\partial \alpha} = -a \frac{\cos\alpha(b + a\cos\alpha) - \sin\alpha(-a\sin\alpha)}{(b + a\cos\alpha)^2} = -a \frac{b\cos\alpha + a\left(\cos^2\alpha + \sin^2\alpha\right)}{(b + a\cos\alpha)^2} = -a \frac{a + b\cos\alpha}{(b + a\cos\alpha)^2} \tag{8}$$

$$\partial_1 \Gamma^1_{22} = \frac{\partial \Gamma^1_{22}}{\partial \alpha} = \frac{b + a\cos\alpha}{a}\cos\alpha - \frac{a\sin\alpha}{a}\sin\alpha = \frac{b + a\cos\alpha}{a}\cos\alpha - \sin^2\alpha \tag{9}$$

Como no hay dependencia con la variable θ las otras derivadas valen 0, ahora sí:

$$(\partial_1 \partial_2 - \partial_2 \partial_1)\, \mathbf{e}_1 = \partial_1(\Gamma^2_{21}\mathbf{e}_2) = (\partial_1 \Gamma^2_{21})\mathbf{e}_2 + \Gamma^2_{21}(\partial_1 \mathbf{e}_2) = \underbrace{\left[(\partial_1 \Gamma^2_{12}) + \Gamma^2_{21}\Gamma^2_{12}\right]}_{R^2{}_{112}} \mathbf{e}_2 \tag{10}$$

$$R^2{}_{112} = -R^2{}_{121} = -a\frac{a + b\cos\alpha}{(b + a\cos\alpha)^2} + \frac{a^2\sin^2\alpha}{(b + a\cos\alpha)^2} = \frac{a^2(\sin^2\alpha - 1) - ab\cos\alpha}{(b + a\cos\alpha)^2} = = -a\cos\alpha \frac{a\cos\alpha + b}{(b + a\cos\alpha)^2} = -\frac{a\cos\alpha}{b + a\cos\alpha} = \frac{b}{b + a\cos\alpha} - 1 \tag{11}$$

$$(\partial_1 \partial_2 - \partial_2 \partial_1)\, \mathbf{e}_2 = \partial_1(\Gamma^1_{22}\mathbf{e}_1) - \partial_2(\Gamma^2_{12}\mathbf{e}_2) = (\partial_1 \Gamma^1_{22})\mathbf{e}_1 - \Gamma^2_{12}\Gamma^1_{22}\mathbf{e}_1 = \underbrace{\left[\partial_1 \Gamma^1_{22} - \Gamma^2_{12}\Gamma^1_{22}\right]}_{R^1{}_{212}} \mathbf{e}_1 \tag{12}$$

$$R^1{}_{212} = -R^1{}_{221} = \frac{b + a\cos\alpha}{a}\cos\alpha - \sin^2\alpha + \sin^2\alpha = \frac{b + a\cos\alpha}{a}\cos\alpha = \cos\alpha\left(\frac{b}{a} + \cos\alpha\right) \tag{13}$$

Utilizando que $R_{\sigma\gamma\alpha\beta} = g_{\sigma\lambda} R^\lambda{}_{\gamma\alpha\beta}$ tenemos la siguiente tabla:

$\mathrm{R}^1{}_{212} = \cos\alpha \frac{b + a\cos\alpha}{a}$	$\mathrm{R}_{1212} = a\cos\alpha(b + a\cos\alpha)$
$\mathrm{R}^2{}_{112} = -\frac{a\cos\alpha}{b + a\cos\alpha}$	$\mathrm{R}_{2112} = -a\cos\alpha(b + a\cos\alpha)$
$\mathrm{R}^1{}_{221} = -\cos\alpha \frac{b + a\cos\alpha}{a}$	$\mathrm{R}_{1221} = -a\cos\alpha(b + a\cos\alpha)$
$\mathrm{R}^2{}_{121} = \frac{a\cos\alpha}{b + a\cos\alpha}$	$\mathrm{R}_{2121} = a\cos\alpha(b + a\cos\alpha)$

Vemos que tal como esperábamos $R_{1212} = -R_{2112}$.

3 Tensor de Ricci

Para calcular el Tensor de Ricci simplemente usaremos la ecuación (16.2) del formulario de Crul

$$R_{jl} = R^k{}_{jkl} \tag{14}$$

Por lo que simplemente con la tabla anterior tenemos que

$$R_{11} = \frac{a\cos\alpha}{b + a\cos\alpha} \tag{15}$$

$$R_{22} = \cos\alpha \frac{b + a\cos\alpha}{a} \tag{16}$$

Siendo todos los demás 0

4 Curvatura escalar

De nuevo, con la ecuación (16.3) del formulario de Crul

$$R = g^{ij} R_{ij} \tag{17}$$

Tenemos que

$$g_{11} = \mathbf{e}_1 \cdot \mathbf{e}_2 = a^2 \Longrightarrow g^{11} = \frac{1}{a^2} \tag{18}$$

$$g_{22} = \mathbf{e}_2 \cdot \mathbf{e}_2 = (b + a\cos\alpha)^2 \Longrightarrow g^{22} = \frac{1}{(b + a\cos\alpha)^2} \tag{19}$$

Por lo tanto

$$R = g^{11} R_{11} + g^{22} R_{22} = \frac{\cos\alpha}{a(b + a\cos\alpha)} + \frac{\cos\alpha}{a(b + a\cos\alpha)} = \frac{2\cos\alpha}{a(b + a\cos\alpha)} \tag{20}$$